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Folding transformations for quantum Painlevé equations

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Abstract
We examine a special property of Painlevé equations, namely possessing folding transformations. The latter are relations of the solution of a given Painlevé equation to the *square* of that of some other, which can be the same as the initial one. They generally exist only for special values of the parameters of a given equation. The present setting will be that of the quantum Painlevé equations, which are systems where the dependent variables are noncommuting objects. Both continuous and discrete cases are analysed and the folding transformations are established in a perfect parallel between continuous and discrete systems.

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1. Introduction

Why do some systems deserve the name of Painlevé equations? The original ones are the second order, nonautonomous, nonlinear differential equations discovered by Painlevé [1] and his group [2]. They were derived using what, in a modern parlance, would qualify as singularity analysis. They were the result of an exhaustive search for equations the solutions of which are devoid of movable critical singularities. This last property, later to be dubbed the Painlevé property [3], can be considered as practically a definition of integrability. (Indeed, in the absence of movable critical singularities, the integration of a nonlinear system should not in principle present more fundamental difficulties than that of a linear one.) The predicted integration of the Painlevé equations was indeed obtained. A first step was realized by Garnier [4] who produced what we call today the Lax pairs for the Painlevé equations, i.e. he showed that the latter result from the compatibility of a system of linear partial differential equations.

The conclusive step was the integration of P_{II} (that of the other Painlevé equations was soon to follow) by Ablowitz and Segur [5], based on inverse scattering techniques.

The integrable character of the Painlevé equations confers them a host of special properties [6]. Among them we can cite the fact that they organize themselves in a degeneration cascade, can be written in a bilinear form, possess Lax pairs, Miura, Bäcklund and Schlesinger transformations, special solutions for particular values of their parameters, their solutions possess contiguity relations (the list being non-exhaustive). It is these special properties of the Painlevé equations that allow us to answer the question we started this paper with. If a system has many (most?) of the properties which characterize the Painlevé equations, then it is reasonable to attribute the name of Painlevé to it. Perhaps the most famous example of 'new' Painlevé systems are the discrete Painlevé equations [7] discovered in the past decades. They are integrable second order, nonautonomous, nonlinear mappings which enjoy practically all the properties of the Painlevé equations [8]. (The fact that their continuous limit is a Painlevé equation is not conclusive *per se*. Discrete systems are more fundamental than continuous ones and the fact that the *continuous* Painlevé equations were discovered first is due to just historical reasons, and to our familiarity with differential rather than difference systems.) Over the years, many more equations of the Painlevé variety were proposed: ultradiscrete [9], delay-differential [10], supersymmetric [11] and quantum (both continuous [12] and discrete [13]), the list being open for future additions. Not all of them share all the properties of the continuous Painlevé equations, but they share enough to warrant the Painlevé name.

This paper will be devoted to what we call quantum Painlevé equations, both continuous and discrete. The term 'quantum' was coined in order to indicate systems where the dependent variables are non-commuting objects. Continuous quantum Painlevé equations have been studied in detail in a series of papers by one of the present authors (HN) [14]. Quantum Painlevé equations are a quantization of Painlevé equations preserving the affine Weyl group symmetries. Since Painlevé equations are Hamiltonian systems the quantization is introduced by replacing the Poisson bracket by a commutator. As such a quantization is not unique the quantum forms of Painlevé equations were determined in such a way that the latter have affine Weyl group actions just as the classical Painlevé equations. The discrete analogues of the quantum Painlevé equations have been the object of studies of the remaining authors [15]. The main difficulty in quantizing discrete systems, integrability notwithstanding, lies in the fact that one must introduce a commutation rule consistent with the evolution. This is a highly nontrivial problem. We have addressed this question in [16] where consistent commutation rules have been proposed. The relation between continuous and discrete quantum Painlevé equations were the object of joint work of the present authors in [17].

The property of Painlevé equations we shall focus on in the present work is that dubbed 'folding' by Okamoto, Sakai and Tsuda [18]. In the Okamoto *et al* terminology, folding transformations are algebraic transformations of the Painlevé systems which give rise to a non-trivial quotient map of the space of initial conditions. In a more elementary way, we can say that folding transformations relate the solution of a given Painlevé equation to the *square* of that of some other one (which can be the same as the initial one) [19]. The fact that such quadratic relations exist can be traced back to the fact that the Painlevé equations have singularities which, in general, can be simple or double poles (or zeros). Thus a quadratic relation relates a solution which has only simple poles to one where all poles are double. These relations generally exist only for special values of the parameters of a given Painlevé equation. (One most interesting result of Okamoto *et al* was the discovery that P_{IV} possesses a folding transformation of degree three, mapping solutions into solutions of the same equation for different values of parameters.) Let us present a simple example of a quadratic folding relation. We start with P_{II}:

$$u'' = 2u^3 + tu + \alpha \tag{1.1}$$

in which we take $\alpha = 0$. Multiplying by u and introducing $w = u^2$ we obtain the equation

$$w'' = \frac{w'^2}{2w} + 4w^2 + 2tw, \tag{1.2}$$

which is equation XX in the Painlevé–Gambier classification [20].

The extension of folding transformations to a discrete setting was presented by some of the present authors in [19]. We have shown that a nice parallel does exist between the properties of the continuum systems and those of their discrete analogues. However, the discrete Painlevé equations may possess some quadratic relations of their own, without reference whatsoever to continuous systems. Let us illustrate this last point with a particular case of the d-P_I equation

$$x_{n+1} + x_{n-1} + x_n = \frac{z_n}{x_n}, \tag{1.3}$$

where $z = \alpha n + \beta$ for some constant α, β. We multiply by x_n both sides of (1.3) and introduce the variables $X = x^2$ and $y_n = x_n x_{n+1}$. We have thus from (1.3), $y_n + y_{n-1} + X_n = z_n$ and, from the definition of y, $X_n X_{n+1} = y_n^2$. Eliminating X between the two equations we obtain for y the mapping

$$(y_{n+1} + y_n - z_{n+1})(y_n + y_{n-1} - z_n) = y_n^2. \tag{1.4}$$

Equation (1.4) is another special form of a d-P_I which was first obtained in [21]. Thus we have established a quadratic relation, which is in fact a degenerate form of a Miura transformation, between two d-P_I's [22].

Before proceeding to the details of the quantum setting, it is interesting to summarize what is known in the commuting case which will serve as a canvas for our presentation. The following quadratic relations exist between continuous Painlevé equations:

$$\begin{bmatrix} i & P_{II}(\alpha = 0) & P_{20} \\ ii & P_{III}(\alpha = \beta = 0) & P_{III}^{(0)}(\gamma = \delta = 0) \\ iii & P_{III}(\alpha = -\beta) & P_{V}(\alpha = \beta = 0) \\ iv & P_{V}(\alpha = -\beta, \gamma = 0) & P_{V}(\delta = 0, \alpha = 0) \\ v & P_{VI}(\alpha = \beta, \gamma = \delta) & P_{VI}(\alpha = \beta = 0) \end{bmatrix},$$

where by $P_{III}^{(0)}$ we denote the zero-parameter Painlevé III [23]. In what follows, we shall present the quantum analogues of relations *i* to *iv* both for continuous and discrete cases. The relation *v* which concerns P_{VI} will be missing from the present paper, since the study of the quantum form of P_{VI} has not been fully completed yet [24]. Moreover, the third degree folding of Okamoto *et al* will not be addressed. As a matter of fact this is a result still missing in the discrete commuting case, despite the efforts of some of the present authors, in collaboration with Sakai. We hope to return to these questions in some future work.

2. Quadratic relations for continuous quantum Painlevé equations

As we have seen in the introduction, for the commuting case a simple quadratic relation is that relating P_{II} to equation XX in the Painlevé–Gambier classification, and which is a limit of equation XXXIV of the same classification, traditionally referred to as P_{34}. Here we shall present this folding for the quantum analogues of the equations. In [13] we have shown that the quantum analogues of P_{II} and P_{34}, two equations related by a Miura transformation, can be written as

$$x'' = 2x^3 - tx + \beta \tag{2.1}$$

and

$$y'' = \tfrac{1}{2}y'y^{-1}y' - 4y^2 + 2ty - \tfrac{1}{2}(\alpha^2 - \hbar^2)y^{-1}, \tag{2.2}$$

where $\beta = 2\alpha - 1$. The commutators of the dependent variables are $[x, x'] = 2\hbar$ and $[y, y'] = 2\hbar y$. The folding transformation exists when $\beta = 0$. In this case, which is the quantum equivalent of *i*, we introduce the variable $w = -x^2$ which leads to the following equation for w:

$$w'' = \tfrac{1}{2}w'w^{-1}w' - 4w^2 - 2tw + 6\hbar^2 w^{-1}. \tag{2.3}$$

This is the quantum form of P_{20}. The commutation relation for the dependent variable is $[w, w'] = 8\hbar w$. It is interesting to cast this relation in the same form as that of P_{34}, i.e. $[w, w'] = 2\tilde{\hbar} w$ which shows that the 'effective' $\hbar$, resulting form the folding is $\tilde{\hbar} = 4\hbar$.

The second folding transformation we are going to examine, corresponding to case *ii* of the introduction is that relating the Painlevé III equation to a different transcendental equation which is known as the 'zero-parameter' P_{III}. We start from the quantum form of P_{III} obtained in [17], given here in a slightly non-canonical form,

$$x'' = x'x^{-1}x' - \frac{x'}{t} + \frac{x^3}{t^2} + \frac{\alpha x^2}{t^2} + \frac{\beta}{t} - x^{-1}. \tag{2.4}$$

The commutation relation of the dependent variable is $[x, x'] = \hbar x^2/t$. Next we introduce the quadratic relation $w = x^2$ and, as in the commuting case, take $\alpha = \beta = 0$. We find

$$w'' = w'w^{-1}w' - \frac{w'}{t} + \frac{2w^2}{t^2} - 2. \tag{2.5}$$

The commutator is now $[w, w'] = 4\hbar w^2/t$. Equation (2.5) is indeed the zero-parameter P_{III} albeit in a non-canonical form. It is elementary to put it in the same form as (2.4), which makes the comparison easier. It suffices to introduce a new independent variable $z = t^2$, with a consequence a commutation relation $[w, w'] = 2\hbar w^2/z$ where the prime now denotes the derivative with respect to z instead of t. We obtain

$$w'' = w'w^{-1}w' - \frac{w'}{z} + \frac{w^2}{2z^2} - \frac{1}{2z}. \tag{2.6}$$

Comparing (2.6) to (2.4) we see that the terms where the dependent variable appears in powers three and minus one are absent while the terms quadratic in the independent variable and constant have fixed coefficients, as should be the case for the zero-parameter P_{III}.

The third transformation, analogue to case *iii*, relates a special case of P_{III} to a special case of Painlevé V. It is more convenient in this case to work with equations where the independent variable appears in an exponential form. Our starting point is the Painlevé III equation in the form

$$x'' = x'x^{-1}x' + e^{2t}(x^3 - x^{-1}) + e^t(\alpha x^2 + \beta) \tag{2.7}$$

with commutator $[x', x] = \hbar x^2$. In order to implement the folding transformation we take $\beta = -\alpha$. In perfect analogy to the commuting case we introduce the change of the variable $w = (x+1)^2(x-1)^{-2}$. A straightforward although rather delicate calculation (requiring the assistance of computer algebra) leads to $\mathrm{P_V}$ in the form

$$w'' = w'\left(\frac{1}{w-1}+\frac{1}{2w}\right)w' - \frac{3\hbar^2}{32}(w-1)^2\left(w-\frac{1}{w}\right) - 4\alpha e^t w + 8e^{2t}\frac{w(w+1)}{(1-w)}. \tag{2.8}$$

The commutation rule for the variable w turns out to be of $[w', w] = \hbar w(w-1)^2$.

The last transformation, analogue to case *iv*, relates two cases of $\mathrm{P_V}$. In the first the parameters satisfy some relationships but still this $\mathrm{P_V}$ is the generic one. The second one (which results from the folding) is a special case of $\mathrm{P_V}$ which is in fact related to $\mathrm{P_{III}}$ through a Miura transformation. Here also, we work with the independent variable in an exponential form. The starting point is the Painlevé V equation in the following form:

$$x'' = x'\left(\frac{1}{x-1}+\frac{1}{2x}\right)x' + \alpha(x-1)^2\left(x-\frac{1}{x}\right) + e^t\frac{x(x+1)}{2(1-x)}. \tag{2.9}$$

We remark that a term proportional to $e^{t/2}x$ is absent and that the two terms with $(x-1)^2$ common factor have opposite coefficients. The commutator is simply $[x', x] = \hbar x(x-1)^2$. Next we introduce the folding transformation $w = 1-(x-1)^2(x+1)^{-2}$. A less straightforward and even more delicate than in case *iii* calculation leads to a new $\mathrm{P_V}$ in the form

$$w'' = w'\left(\frac{1}{w-1}+\frac{1}{2w}\right)w' + (w-1)^2\left(\frac{5\hbar^2}{2}w - \frac{\alpha}{w}\right) + e^t w. \tag{2.10}$$

The commutation rule for the variable w turns out to be $[w', w] = -4\hbar w(w-1)^2$. The absence of a term of the form $e^{2t}w(w+1)/(1-w)$ indicates that this special form of $\mathrm{P_V}$ is indeed a Miura transformed $\mathrm{P_{III}}$. We may remark here that the coefficient of the $(w-1)^2 w$ term is proportional to $\hbar^2$: this is consistent with the fact that this term is absent in the commutative case.

3. Quadratic relations for discrete quantum Painlevé equations

The first case we are going to examine here is the discrete analogue of case *i*, namely the folding transformation between the discrete $\mathrm{P_{II}}$ and the discrete form of $\mathrm{P_{20}}$. In what follows the independent variable will be introduced through $z_n \equiv \delta(n-n_0)$. We start with the 'standard' symmetric (in the QRT [25] sense) form of d-$\mathrm{P_{II}}$

$$x_{n+1} + x_{n-1} = \frac{z_n x_n}{x_n^2 - 1} \tag{3.1}$$

where we have taken the parameter which appears in the numerator of d-$\mathrm{P_{II}}$ to zero (for the folding transformation to exist). The quantum form of d-$\mathrm{P_{II}}$ was first obtained in [26]. The commutation relation of the dependent variables is $[x_{n+1}, x_n] = \hbar$. Next we introduce the following folding transformtion. As in the commuting case, two variables are necessary. We put $y_n = x_n^2$ and for the second variable we distinguish the even and odd indices: $w_{2n} = x_{2n}x_{2n+1} - \hbar/2$ and $w_{2n-1} = x_{2n}x_{2n-1} + \hbar/2$. (We should point out here that the $\hbar$ shift is *not* one that could be absorbed by a simple symmetrization.) We find

$$w_n + w_{n-1} = \frac{z_n y_n}{y_n - 1} \tag{3.2a}$$

complemented by

$$y_{2n}y_{2n+1} = w_{2n}^2 - \hbar^2/4 \tag{3.2b}$$

$$y_{2n}y_{2n-1} = w_{2n-1}^2 - \hbar^2/4. \tag{3.2c}$$

Given the form of (3.2*a*) it is possible to solve for y and thus eliminate it completely from the equation. We obtain

$$\left(\frac{w_{2n}+w_{2n-1}}{w_{2n}+w_{2n-1}-z_{2n}}\right)\left(\frac{w_{2n+1}+w_{2n}}{w_{2n+1}+w_{2n}-z_{2n+1}}\right)=w_{2n}^2-\hbar^2/4 \tag{3.3}$$

and

$$\left(\frac{w_{2n}+w_{2n-1}}{w_{2n}+w_{2n-1}-z_{2n}}\right)\left(\frac{w_{2n-1}+w_{2n-2}}{w_{2n-1}+w_{2n-2}-z_{2n-1}}\right)=w_{2n-1}^2-\hbar^2/4. \tag{3.4}$$

Equation (3.3) is a quantum form of d-P$_{20}$. The commutator of the w variables is $[w_{n-1}, w_n] = 2\hbar(w_n + w_{n-1})(w_n + w_{n-1} - z_n)/z_n$.

At this stage it is interesting to relate the quantum form of d-P$_{20}$ (which we recall is a special form of d-P$_{34}$) to a (quantum form of) d-P$_{\mathrm{II}}$. We start by introducing the variable u and again distinguish the even and odd indices. We have $u_{2n} = y_{2n}^{-1}(w_{2n} - \hbar/2)$ and $u_{2n+1} = y_{2n+2}^{-1}(w_{2n+1} + \hbar/2)$. Given these definitions u has a very simple expression in terms of the variable of d-P$_{\mathrm{II}}$: $u_{2n} = x_{2n}^{-1}x_{2n+1}$ and $u_{2n+1} = x_{2n+2}^{-1}x_{2n+1}$. After a lengthy calculation we obtain the following equation:

$$z_n(u_n+u_{n-1})^{-1}+z_{n+1}(u_n+u_{n+1})^{-1}=u_n^{-2}+\left(z_n+\frac{\delta}{2}(1+(-1)^n)+(-1)^n\hbar\right)u_n^{-1}-1, \tag{3.5}$$

which is the quantum form of what is usually called the 'alternate' d-P$_{\mathrm{II}}$ in the form we obtained in [27]. (The expression $(z_n + \delta(1 + (-1)^n)/2)$ in the rhs of (3.5) means that the independent variable dependence is z_{n+1} when n is even and z_n when n is odd.) We complement this analysis by the commutator of the variables u: $[u_{n-1}, u_n] = 2\hbar(u_{n-1} + u_n)^2/z_n$. Given the form of this commutator we remark that u_n does indeed commute with the lhs of (3.5).

The next case, which is the discrete analogue of *ii*, is a folding transformation on the quantum discrete form of Painlevé III. We start from the q-discrete P$_{\mathrm{III}}$:

$$x_{n+1}x_{n-1}=q\frac{(x_n-az_n)(x_n-bz_n)}{(1-cx_n)(1-dx_n)}, \tag{3.6}$$

where $z_n = z_0\lambda^n$. We have shown in [15] that the consistent commutation relation here is $x_{n+1}x_n = qx_nx_{n+1}$. As in the commuting case the folding transformation requires two constraints $b = -a$ and $d = -c$, leading to

$$x_{n+1}x_{n-1}=q\frac{x_n^2-a^2z_n^2}{1-c^2x_n^2}. \tag{3.7}$$

We can now introduce $w = x^2$ and obtain for w the equation

$$w_{n+1}w_{n-1}=q^4\frac{\left(w_n-a^2z_n^2\right)\left(w_n-a^2z_n^2/q^2\right)}{(1-c^2w_n)(1-c^2q^2w_n)}. \tag{3.8}$$

The commmutation relation is now $w_{n+1}w_n = q^4w_nw_{n+1}$ which is consistent with the presence of q^4 in the rhs of (3.8).

Since the case we treat here is the discrete analogue of case *ii* we expect the folding transformation to relate a special case of P$_{\mathrm{III}}$ to the zero-parameter P$_{\mathrm{III}}$. However *a priori* nothing distinguishes (3.7) from (3.8). The answer to this paradox can be found in the continuous limit. In order to obtain the full continuous P$_{\mathrm{III}}$ from (3.6) we must introduce a small parameter ϵ, put $q = 1 + \epsilon\hbar$, $a = \gamma\epsilon + \alpha\epsilon^2$, $b = -\gamma\epsilon + \alpha\epsilon^2$, $c = \delta\epsilon + \beta\epsilon^2$, $d = -\delta\epsilon + \beta\epsilon^2$ and take the limit $\epsilon \to 0$. We obtain for x the continuous P$_{\mathrm{III}}$ involving four parameters, α, β, γ and δ (and the standard form of P$_{\mathrm{III}}$ corresponds to taking $\gamma = \delta = 1$). The folding transformation requires $\alpha = \beta = 0$ and (3.7) is compatible with this while keeping $\gamma = \delta = 1$.

However, for (3.8) the continuous limit (where $q \to 1$) is only possible for $\gamma = \delta = 0$ which precisely corresponds to the zero-parameter $\mathrm{P_{III}}$.

The third folding, *iii*, relates a special case of Painlevé III to a special case of Painlevé V. In analogy to the continuous case we shall start from a $\mathrm{P_{III}}$ which belongs to the family of what we have dubbed in [28] the 'master d-$\mathrm{P_{II}}$' equation. Its form involving commuting objects is

$$\frac{z_{n+1}}{1-x_nx_{n+1}} + \frac{z_n}{1-x_nx_{n-1}} = \frac{z_{n+1/2}+a}{1-x_n^2/t} + \frac{z_{n+1/2}-a}{1-tx_n^2}, \tag{3.9}$$

which can be split into the system

$$y_n + y_{n+1} = \frac{z_{n+1/2}+a}{1-x_n^2/t} + \frac{z_{n+1/2}-a}{1-tx_n^2} \tag{3.10a}$$

$$x_nx_{n-1} = \frac{y_n - z_n}{y_n}. \tag{3.10b}$$

We know that x and y are noncommuting objects and introduce the commutators (following our results in [16]) $[x_n, y_n] = \hbar x_n$ and $[x_{n-1}, y_n] = -\hbar x_{n-1}$. The quantum form of (3.10) involves explicit quantum corrections

$$y_n + y_{n+1} = \frac{z_{n+1/2}+a}{1-x_n^2/t} + \frac{z_{n+1/2}-a}{1-tx_n^2} + \hbar \tag{3.11a}$$

$$x_nx_{n-1} = \frac{y_n - z_n - \hbar/2}{y_n - \hbar/2}. \tag{3.11b}$$

From (3.11b) we obtain, with the appropriate use of commutators, the equation

$$x_n^2x_{n-1}^2 = \frac{(y_n - z_n)^2 - \hbar^2/4}{y_n^2 - \hbar^2/4}. \tag{3.12}$$

The folding transformation is now simply $w = x^2$, which leads to the system

$$y_n + y_{n+1} = \frac{z_{n+1/2}+a}{1-w_n/t} + \frac{z_{n+1/2}-a}{1-tw_n} + \hbar \tag{3.13a}$$

$$w_nw_{n-1} = \frac{(y_n - z_n)^2 - \hbar^2/4}{y_n^2 - \hbar^2/4}. \tag{3.13b}$$

System (3.13) is the quantum form of (a special case of) an equation obtained in [29] and which is a discrete analogue of Painlevé V. Thus we have established the quantum analogue of the discrete case *iii* folding. The commutator of w with y can be calculated in a straightforward way. We find $[w_n, y_n] = 2\hbar w_n$ and $[w_{n-1}, y_n] = -2\hbar w_{n-1}$.

The final folding case is the one relating two forms of $\mathrm{P_V}$, i.e. the quantum discrete analogue of case *iv*. Our starting point is a q-discrete Painlevé V which in the noncommuting case can be written as [16]:

$$(x_{n+1}x_n - 1)(x_nx_{n-1} - 1) = q\frac{1 + \alpha x_n + \beta x_n^2 + q\alpha x_n^3 + q^2x_n^4}{1 + \gamma z_nx_n + \delta z_n^2x_n^2}, \tag{3.14}$$

where $z_n = z_0\lambda^n$. The commutation relation of the dependent variable is $x_{n+1}x_n = qx_nx_{n+1} + 1 - q$ which can be rewritten as $(x_{n+1}x_n - 1) = q(x_nx_{n+1} - 1)$. In order to implement the folding transformation we take $\alpha = \gamma = 0$ and introduce the variable $y = x^2$.

Moreover an auxiliary variable is necessary at this point and we introduce $w_n = x_n x_{n-1} - 1$. We can now split (3.14) into

$$w_{n+1}w_n = q\frac{q^2y_n^2 + \beta y_n + 1}{1 + \delta z_n^2 y_n} \tag{3.15a}$$

and a second equation obtained in an elementary way

$$y_n y_{n-1} = (w_n + 1)(qw_n + 1). \tag{3.15b}$$

We can easily check that the commutation relations are $y_n w_n = q^2 w_n y_n$ and $w_n y_{n-1} = q^2 y_{n-1} w_n$. It is interesting at this stage to rescale y introducing $v = y\sqrt{q}$. The commutation relations between w and v are the same as those for w and y. The equations are now

$$w_{n+1}w_n = q^2\frac{v_n^2 + \tilde{\beta} v_n + 1}{1 + \tilde{\delta} z_n^2 v_n} \tag{3.16a}$$

$$v_n v_{n-1} = q^2(w_n + 1)(w_n + 1/q), \tag{3.16b}$$

where $\tilde{\beta} = \beta q^{-3/2}$ and $\tilde{\delta} = \delta q^{-1/2}$. System (3.16) is the appropriate quantum form of (a reduced case of) asymmetric q-$\mathrm{P_{III}}$. Because of the absence of a denominator in (3.16*b*) the continuous limit of this asymmetric q-$\mathrm{P_{III}}$ is not $\mathrm{P_{VI}}$ [30] but $\mathrm{P_V}$ and, in fact, a $\mathrm{P_V}$ with one missing parameter. This is precisely the $\mathrm{P_V}$ which is the Miura transformed $\mathrm{P_{III}}$. Moreover, when we go to the continuous limit, which implies $q \to 1$, one more parameter is absent in the resulting $\mathrm{P_V}$.

4. Conclusions

In this paper we have studied the folding transformations which exist for continuous and discrete quantum Painlevé equations. We have established a perfect parallel between the transformations which exist for the 'standard' commuting Painlevé equations and their noncommuting, quantum, analogues. As we have explained in previous publications of ours, the discrete domain presents additional difficulties. Not only must one introduce the proper commutation rules which have to be compatible with the discrete evolution but one must also choose the proper discrete systems which will be related by the folding transformation. This is due to the fact that the continuous limit of some discrete equation is too strong a reduction and one cannot base predictions on this feature alone. Thus, for instance, the folding transformation which for continuous systems exists between $\mathrm{P_{III}}$ and the zero-parameter $\mathrm{P_{III}}$ has as discrete analogue a folding transformation between two different $\mathrm{P_{III}}$ s: only at the continuous limit is the distinction between full $\mathrm{P_{III}}$ and zero-parameter $\mathrm{P_{III}}$ established.

While we have striven to make this paper as complete as possible, it is clear that the question of folding transformations for quantum Painlevé equations has not been exhausted. In particular it will be interesting to study the folding transformations on the quantum discrete and continuous forms of $\mathrm{P_{VI}}$. However, this is not immediately possible given the present state of our knowledge since the study of the quantum continuous $\mathrm{P_{VI}}$ is not yet complete, to say nothing of that of its discrete analogue. Another most interesting result would be the derivation of the folding transformation for the Painlevé IV equation, which in the continuous case was shown by Okamoto and collaborators to be of third degree. Unfortunately, the result is missing already for the discrete system even in the commuting case and thus the extension to the noncommuting setting will have to wait for some progress. We hope to be able to address these questions in some future work of ours.

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